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Novikov algebras with associative bilinear forms

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Abstract
Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic-type and Hamiltonian operators in formal variational calculus. The goal of this paper is to study Novikov algebras with non-degenerate associative symmetric bilinear forms, which we call quadratic Novikov algebras. Based on the classification of solvable quadratic Lie algebras of dimension not greater than 4 and Novikov algebras in dimension 3, we show that quadratic Novikov algebras up to dimension 4 are commutative. Furthermore, we obtain the classification of transitive quadratic Novikov algebras in dimension 4. But we find that not every quadratic Novikov algebra is commutative and give a non-commutative quadratic Novikov algebra in dimension 6.

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1. Introduction

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic-type [1–3] and Hamiltonian operators in the formal variational calculus [4–9]. A Novikov algebra A is a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \mapsto xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \tag{1}$$

$$(xy)z = (xz)y, \tag{2}$$

for any $x, y, z \in A$. The commutator of A,

$$[x, y] = xy - yx, \tag{3}$$

defines a Lie algebra $\mathrm{g} = \mathrm{g}(A)$, which is called the sub-adjacent Lie algebra of A. Novikov algebras are a special class of left symmetric algebras which only satisfy equation (1). Left symmetric algebras are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [10–14].

A bilinear form $f : A \times A \to \mathbb{F}$ is associative if and only if

$$f(xy, z) = f(x, yz), \qquad \forall x, y, z \in A. \tag{4}$$

The goal of this paper is to study the pair (A, f), where A denotes a Novikov algebra and f denotes a non-degenerate associative symmetric bilinear form of A. In abuse of notation we will also use the term quadratic Novikov algebra for denoting such a pair. The motivation for studying quadratic Novikov algebras comes from the fact that Lie or associative algebras with forms have important applications in several areas of mathematics and physics, such as the structure theory of finite-dimensional semi-simple Lie algebras, the theory of complete integrable Hamiltonian systems and the classification of statistical models over two-dimensional graphs.

There has been a lot of progress in the study of Novikov algebras [15–25]. It is given in [26] that quadratic Novikov algebras are associative. In this paper, we show that quadratic Novikov algebras up to dimension 4 are commutative, although the classification of Novikov algebras in dimension 4 is not completely given. And we also show that not every quadratic Novikov algebra is commutative.

The paper is organized as follows. In section 2, we show that $(\mathrm{g}(A), f)$ is a quadratic Lie algebra, where (A, f) denotes a quadratic Novikov algebra and $\mathrm{g}(A)$ denotes the sub-adjacent Lie algebra of A. In section 3, we give the classification of solvable quadratic Lie algebras up to dimension 4. In section 4, based on the classification of solvable quadratic Lie algebras of dimension not greater than 4 and Novikov algebras in dimension 3, we show that quadratic Novikov algebras up to dimension 4 are commutative. In section 5, we give the classification of transitive quadratic Novikov algebras in dimension 4. In section 6, we show that not every quadratic Novikov algebra is commutative. In section 7, we list some conclusions.

Throughout this paper we assume that the algebras are of finite dimension over $\mathbb{C}$.

2. Some properties on quadratic Novikov algebras

Suppose that (A, f) is a quadratic Novikov algebra. Let H be a subspace of A and $\mathrm{g}(A)$ be the sub-adjacent Lie algebra of A. Let $H^{\perp} = \{x \in A \mid f(x, y) = 0, \forall y \in H\}$ and $Z(A) = \{x \in A \mid xy = yx = 0, \forall y \in A\}$. It is easy to see that

$$Z(A) = (AA)^{\perp}, \tag{5}$$

which implies that

$$\dim Z(A) + \dim AA = \dim A. \tag{6}$$

The pair (g, f) is called a quadratic Lie algebra if g is a Lie algebra and f is a non-degenerate symmetric bilinear form of g satisfying

$$f(x, [y, z]) = f([x, y], z), \qquad \forall x, y, z \in \mathrm{g}. \tag{7}$$

Note that for a quadratic Lie algebra, the adjoint representation is equivalent to the co-adjoint representation. Such Lie algebras are called symmetric self-dual Lie algebras by physicists (e.g. [27]). References [27–29] show the importance of quadratic Lie algebras in conformal field theory.

Proposition 1. *Let (A, f) be a quadratic Novikov algebra and $\mathrm{g}(A)$ be the sub-adjacent Lie algebra of A. Then $(\mathrm{g}(A), f)$ is a quadratic Lie algebra.*

Proof. Since (A, f) is a quadratic Novikov algebra, then

$$\begin{aligned} f([x, y], z) &= f(xy, z) - f(yx, z) \\ &= f(x, yz) - f(zy, x) \\ &= f(x, [y, z]) \end{aligned}$$

for any $x, y, z \in A$. It follows that $(\mathrm{g}(A), f)$ is a quadratic Lie algebra. □

There has been a lot of progress in the study of quadratic Lie algebras. Based on proposition 1, we can get some properties on quadratic Novikov algebras by the study on quadratic Lie algebras.

3. Quadratic Lie algebras

Definition 1. *Let* (g, f) *be a quadratic Lie algebra and* H *be an ideal of* g. *H is called non-degenerate if* $f|_{H\times H}$ *is non-degenerate. H is called isotropic if* $f|_{H\times H} = 0$.

Proposition 2. *Let* (g, f) *be a quadratic Lie algebra. Then we have the following.*

(1) $C(\mathrm{g}) = [\mathrm{g}, \mathrm{g}]^{\perp}$.
(2) Let H be an ideal of g. *Then $H^{\perp}$ is an ideal of* g. *Furthermore, assume that H is non-degenerate. Then $H^{\perp}$ is also non-degenerate and* $\mathrm{g} = H \oplus H^{\perp}$.

As a sequence of proposition 2, we have

$$\dim C(\mathrm{g}) + \dim[\mathrm{g}, \mathrm{g}] = \dim \mathrm{g}. \tag{8}$$

If g is solvable, then $[\mathrm{g}, \mathrm{g}] \neq \mathrm{g}$, which implies that $\dim C(\mathrm{g}) \geqslant 1$. In the following, we will discuss the classification of solvable quadratic Lie algebras (g, f) up to dimension 4 case by case.

Case 1. $\dim \mathrm{g} = 1$.
g is Abelian and there exists a basis $\{e\}$ of g such that $f(e, e) = 1$.

Case 2. $\dim \mathrm{g} = 2$.
There are two cases as follows.

(1) If $\dim C(\mathrm{g}) = 2$, then g is Abelian and there exists a basis $\{e_1, e_2\}$ of g such that

$$f(e_1, e_1) = f(e_2, e_2) = 1$$

and others zero.
(2) If $\dim C(\mathrm{g}) = 1$, we must have $[\mathrm{g}, \mathrm{g}] = 0$, which contradict to $\dim[\mathrm{g}, \mathrm{g}] = 1$.

Case 3. $\dim \mathrm{g} = 3$.
There are two cases as follows.

(1) If $C(\mathrm{g})$ is not isotropic, then there exists an element $x \in C(\mathrm{g})$ such that $f(x, x) \neq 0$. By proposition 2,

$$\mathrm{g} = \mathbb{C}\mathrm{x} \oplus \mathrm{x}^{\perp},$$

where $x^{\perp}$ is a solvable quadratic Lie algebra in dimension 2. It follows that g is Abelian.
(2) If $C(\mathrm{g})$ is isotropic, then $\dim C(\mathrm{g}) = 1$. We must have $\dim[\mathrm{g}, \mathrm{g}] \leqslant 1$, which contradicts to $\dim[\mathrm{g}, \mathrm{g}] = 2$.

Case 4. $\dim \mathrm{g} = 4$.
There are two cases as follows.

(1) If $C(\mathrm{g})$ is not isotropic, then there exists an element $x \in C(\mathrm{g})$ such that $f(x, x) \neq 0$. By proposition 2,

$$\mathrm{g} = \mathbb{C}\mathrm{x} \oplus \mathrm{x}^{\perp},$$

where $x^{\perp}$ is a solvable quadratic Lie algebra in dimension 3. It follows that g is Abelian.

(2) If $C(\mathrm{g})$ is isotropic, then $\dim C(\mathrm{g}) = 1$ or 2.

   (1) If $\dim C(\mathrm{g}) = 2$, we must have $\dim[\mathrm{g}, \mathrm{g}] \leqslant 1$, which contradicts to $\dim[\mathrm{g}, \mathrm{g}] = 2$.

   (2) If $\dim C(\mathrm{g}) = 1$, then $\dim[\mathrm{g}, \mathrm{g}] = 3$ and there exists a basis $\{e_1, e_2, e_3, e_4\}$ of g such that $\{e_4\}$ is a basis of $C(\mathrm{g})$ and $\{e_2, e_3, e_4\}$ is a basis of $[\mathrm{g}, \mathrm{g}]$ and $f(e_1, e_4) = f(e_4, e_1) = f(e_2, e_3) = f(e_3, e_2) = 1$ and others zero. Since $f([e_2, e_3], e_2) = f([e_2, e_3], e_3) = 0$,

$$[e_2, e_3] = ke_4, \tag{9}$$

   where $k \neq 0$ since $\dim[\mathrm{g}, \mathrm{g}] = 3$. Take $k = 1$. Since $f([e_1, e_2], e_2) = f([e_1, e_2], e_1) = 0$ and $f([e_1, e_3], e_3) = f([e_1, e_3], e_1) = 0$, then

$$[e_1, e_2] = ae_2, [e_1, e_3] = be_3. \tag{10}$$

   It is easy to show that

$$a = f(ae_2, e_3) = f([e_1, e_2], e_3) = f(e_1, [e_2, e_3]) = 1$$
$$b = f(be_3, e_2) = f([e_1, e_3], e_2) = f(e_1, [e_3, e_2]) = -1.$$

Therefore, we have

Theorem 1. *The following is the classification of solvable quadratic Lie algebras up to dimension 4.*

(1) If $\dim \mathrm{g} = 1$*, then* g *is Abelian and there exists a basis* $\{e_1\}$ *such that* $f(e_1, e_1) = 1$.

(2) If $\dim \mathrm{g} = 2$*, then* g *is Abelian and there exists a basis* $\{e_1, e_2\}$ *such that* $f(e_1, e_1) = f(e_2, e_2) = 1$ *and others zero.*

(3) If $\dim \mathrm{g} = 3$*, then* g *is Abelian and there exists a basis* $\{e_1, e_2, e_3\}$ *such that* $f(e_1, e_1) = f(e_2, e_2) = f(e_3, e_3) = 1$ *and others zero.*

(4) Solvable quadratic Lie algebras in dimension 4 are one of the following cases.

   (1) g *is Abelian and there exists a basis* $\{e_1, e_2, e_3, e_4\}$ *such that* $f(e_1, e_1) = f(e_2, e_2) = f(e_3, e_3) = f(e_4, e_4) = 1$ *and others zero.*

   (2) There exists a basis $\{e_1, e_2, e_3, e_4\}$ *of* g *such that* $[e_1, e_2] = e_2, [e_1, e_3] = -e_3, [e_2, e_3] = e_4$ *and* $f(e_1, e_4) = f(e_4, e_1) = f(e_2, e_3) = f(e_3, e_2) = 1$ *and others zero.*

4. Quadratic Novikov algebras up to dimension 4

Thanks to theorem 1, quadratic Novikov algebras up to dimension 3 are commutative. If (A, f) is a commutative quadratic Novikov algebra, then

$$f(xy, z) = f(x, yz) = f(x, zy)$$

for any $x, y, z \in A$. Non-degenerate symmetric bilinear forms satisfying

$$f(xy, z) = f(x, zy), \qquad \forall x, y, z \in A \tag{11}$$

on Novikov algebras up to dimension 3 are given in [25].

Let (A, f) be a non-commutative quadratic Novikov algebra in dimension 4. It follows that $(\mathrm{g}(A), f)$ is the very last part (b) of theorem 1. Then we must have

$$\dim AA \geqslant 3, \tag{12}$$

since $[\mathrm{g}(A), \mathrm{g}(A)] \subset AA \subset A$. It is easy to check that the list of Lie ideals is given by $\{e_4\}, \{e_2, e_4\}, \{e_3, e_4\}, \{e_2, e_3, e_4\}, \{e_1, e_2, e_3, e_4\}$.

It is well known that a finite-dimensional Novikov algebra contains a largest transitive ideal $N(A)$ and the quotient algebra $A/N(A)$ is a direct sum of fields. Since the ideal $N(A)$ is naturally a Lie ideal, then $N(A)$ must be one of the above cases. If $\dim N(A) = 1$, then $A/N(A) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. If $\dim N(A) = 2$, then $A/N(A) = \mathbb{C} \oplus \mathbb{C}$. Each of these cases is impossible since $\dim[\mathrm{g}(A), \mathrm{g}(A)] = 3$.

4.1. $N(A) = \{e_2, e_3, e_4\}$

The sub-adjacent Lie algebra of $N(A)$ is Heisenberg and $e_1e_1 = ke_1 + e$, where $k \neq 0$ and $e \in N(A)$. By the classification of transitive Novikov algebras in dimension 3, there exists another basis of $N(A)$, which is also denoted by $\{e_2, e_3, e_4\}$, satisfying one of the following four cases:

(1) $e_2e_3 = e_4, e_3e_2 = -e_4$.
(2) $e_2e_2 = e_4, e_2e_3 = e_4, e_3e_2 = -e_4, e_3e_3 = le_4$.
(3) $e_2e_3 = e_4, e_3e_2 = le_4, e_3e_3 = e_2, l \neq 1$.
(4) $e_3e_2 = e_4, e_3e_3 = e_2$.

For any case, $[e_2, e_3] = me_4, [e_2, e_4] = [e_3, e_4] = 0$, where $m \neq 0$, which implies that

$$f(e_4, x) = 0, \qquad \forall x \in N(A). \tag{13}$$

Since f is non-degenerate, then $e_4 \in C(\mathrm{g}(A))$ and $f(e_4, e_1) \neq 0$. Also we have $xe_4 = e_4x = 0$ for any $x \in N(A)$, which implies

$$f(e_1e_4, e_2) = f(e_1e_4, e_3) = f(e_1e_4, e_4) = 0. \tag{14}$$

It is easy to show that

$$\begin{aligned} mf(e_1e_4, e_1) &= f(e_1, [e_2, e_3]e_1) = f(e_1, e_2(e_3e_1) - e_3(e_2e_1)) \\ &= f((e_1e_2)e_3 - (e_1e_3)e_2, e_1) = 0. \end{aligned}$$

Since $m \neq 0$ and f is non-degenerate, then $e_1e_4 = 0$, i.e.,

$$xe_4 = e_4x = 0, \qquad \forall x \in A, \tag{15}$$

since $e_4 \in C(\mathrm{g}(A))$. The equation (15) implies that

$$\dim AA = 3, \tag{16}$$

which contradicts to

$$\dim AA = 4, \tag{17}$$

since $k \neq 0$ and $\dim[\mathrm{g}(A), \mathrm{g}(A)] = 3$.

4.2. $N(A) = A$

It follows that $\dim AA = 3$, which implies that $\dim Z(A) = 1$. But $Z(A) \subset C(\mathrm{g}(A))$, so we have

$$Z(A) = \{e_4\}. \tag{18}$$

Let H be the vector space spanned by $\{e_1, e_2, e_3\}$, then $A = H + \mathbb{C}e_4$ is a direct sum of subspaces. Let ρ be the projection from A to H. Define a bilinear product on H by

$$e_i \cdot e_j = \rho(e_i e_j), \qquad \forall i, j = 1, 2, 3. \tag{19}$$

It is easy to show that $(H, \cdot)$ is a transitive Novikov algebra and the sub-adjacent Lie algebra $\mathrm{g}(H)$ satisfies

$$[e_1, e_2]_H = e_1 \cdot e_2 - e_2 \cdot e_1 = e_2, [e_1, e_3]_H = -e_3, [e_2, e_3]_H = 0.$$

By the classification of transitive Novikov algebras in dimension 3, there exists another basis of H, which is also denoted by $\{e_1, e_2, e_3\}$, satisfying

$$e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = -e_3 \tag{20}$$

and otherwise zero. It follows that

$$e_1 e_2 = e_2 + a e_4, e_3 e_1 = b e_4, \tag{21}$$

which implies that

$$f(e_3, e_2) = f(e_3, e_1 e_2) = f(e_3 e_1, e_2) = 0,$$

i.e., f is degenerate since $f([e_1, e_3], e_1) = f([e_1, e_3], e_3) = 0$ and $e_4 \in Z(A)$.

Therefore we have the following.

Theorem 2. *Quadratic Novikov algebras up to dimension 4 are commutative.*

5. The classification of transitive quadratic Novikov algebras in dimension 4

The goal of this section is to obtain the classification of transitive quadratic Novikov algebras in dimension 4. We will do it as follows:

(1) obtain the classification of commutative transitive Novikov algebras;

(2) compute bilinear forms of Novikov algebras given by (1).

First of all, we list some notations. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of A, then

$$f(e_i e_j, e_k) = f(e_i, e_j e_k).$$

Moreover, a bilinear form of A under the basis $\{e_1, e_2, \ldots, e_n\}$ is completely decided by the matric $F = (f_{ij})$, where

$$f_{ij} = f(e_i, e_j).$$

f is non-degenerate if and only if $\det F \neq 0$. Let $\left\{c_{ij}^k\right\}$ be the set of structure constants of A, i.e.,

$$e_i e_j = \sum_k c_{ij}^k e_k.$$

Denote the (form)character matrix of a Novikov algebra by

$$\begin{pmatrix} \sum_k c_{11}^k e_k & \cdots & \sum_k c_{1n}^k e_k \\ \vdots & \ddots & \vdots \\ \sum_k c_{n1}^k e_k & \cdots & \sum_k c_{nn}^k e_k \end{pmatrix}.$$

The classification of commutative transitive Novikov algebras over $\mathbb{R}$ in dimension 4 is given in [21]. Then it is easy to show

Proposition 3. *Commutative transitive Novikov algebras over $\mathbb{C}$ in dimension 4 are given as follows:*

Type	Character matrix	Type	Character matrix
$(A1)$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	$(A2)$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&e_1\end{pmatrix}$
$(A3)$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&0\\0&0&0&e_2\end{pmatrix}$	$(A4)$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&e_2\\0&0&e_2&0\end{pmatrix}$
$(A5)$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&e_1\\0&0&e_1&e_3\end{pmatrix}$	$(A6)$	$\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_1&0\\0&0&0&e_1\end{pmatrix}$
$(A7)$	$\begin{pmatrix}0&0&0&0\\0&e_1&0&0\\0&0&e_1&0\\0&0&0&e_1\end{pmatrix}$	$(A8)$	$\begin{pmatrix}0&0&0&0\\0&0&0&e_1\\0&0&e_1&0\\0&e_1&0&e_2\end{pmatrix}$
$(A9)$	$\begin{pmatrix}0&0&0&0\\0&0&0&e_1\\0&0&e_1&e_2\\0&e_1&e_2&e_3\end{pmatrix}$		

Example 1. Non-degenerate associative symmetric bilinear forms of $(A7)$.

By the character matrix, we have $f(e_1, e_1) = f(e_2e_2, e_1) = f(e_2, e_2e_1) = 0$, $f(e_1, e_3) = f(e_2e_2, e_3) = f(e_2, e_2e_3) = 0$, $f(e_1, e_4) = f(e_2e_2, e_4) = f(e_2, e_2e_4) = 0$, $f(e_1, e_2) = f(e_3e_3, e_2) = f(e_3, e_3e_2) = 0$. It follows that $\det F = 0$. Namely, there does not exist such f that $((A7), f)$ is a quadratic Novikov algebra.

Similar to example 1, we have

Proposition 4. *There are not non-degenerate associative symmetric bilinear forms on Novikov algebras* $(A6)$–$(A9)$*. Transitive quadratic Novikov algebras are given as follows, where* $\det F \neq 0$.

Type	Non-degenerate associative symmetric bilinear form	Type	Non-degenerate associative symmetric bilinear form
$(A1)$	$F = \begin{pmatrix}f_{11}&f_{12}&f_{13}&f_{14}\\f_{12}&f_{22}&f_{23}&f_{24}\\f_{13}&f_{23}&f_{33}&f_{34}\\f_{14}&f_{24}&f_{34}&f_{44}\end{pmatrix}$	$(A2)$	$F = \begin{pmatrix}0&0&0&f_{14}\\0&f_{22}&f_{23}&f_{24}\\0&f_{23}&f_{33}&f_{34}\\f_{14}&f_{24}&f_{34}&f_{44}\end{pmatrix}$
$(A3)$	$F = \begin{pmatrix}0&0&f_{13}&0\\0&0&0&f_{24}\\f_{13}&0&f_{33}&f_{34}\\0&f_{24}&f_{34}&f_{44}\end{pmatrix}$	$(A4)$	$F = \begin{pmatrix}0&0&f_{13}&f_{14}\\0&0&f_{14}&0\\f_{13}&f_{14}&f_{33}&f_{34}\\f_{14}&0&f_{34}&f_{44}\end{pmatrix}$
$(A5)$	$F = \begin{pmatrix}0&0&0&f_{14}\\0&f_{22}&0&f_{24}\\0&0&f_{14}&f_{34}\\f_{14}&f_{24}&f_{34}&f_{44}\end{pmatrix}$		

6. A non-commutative quadratic Novikov algebra

A natural problem is whether every quadratic Novikov algebra is commutative. We find that there exist non-commutative quadratic Novikov algebras in higher dimensions. The following is an example in dimension 6.

Let A be a vector space with a basis $\{e_1, e_2, \ldots, e_6\}$. Define a bilinear product on A under the basis $\{e_1, e_2, \ldots, e_6\}$ by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_2 & 0 & 0 \end{pmatrix}$$

and a bilinear form f of A under the basis $\{e_1, e_2, \ldots, e_n\}$ by the matrix

$$F = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then A is a Novikov algebra since

$$(xy)z = x(yz) = 0, \qquad \forall x, y, z \in A. \tag{22}$$

It is easy to check that f is a non-degenerate associative symmetric bilinear form of A. But A is non-commutative.

7. Conclusions

Since the classification of Novikov algebras in higher dimensions, even in dimension 4, is also unsolved, the direct study on quadratic Novikov algebras is difficult. A new idea is given in this paper, and we get some interesting results. It is reasonable to believe that much more and better results can be obtained by this way.

The following are some conclusions on quadratic Novikov algebras according to the discussion in the previous sections.

(1) $(\mathrm{g}(A), f)$ is a quadratic Lie algebra, where (A, f) denotes a quadratic Novikov algebra and $\mathrm{g}(A)$ denotes the sub-adjacent Lie algebra of A.
(2) Quadratic Novikov algebras up to dimension 4 are commutative. And it is easy to check that they are also associative.
(3) There are not non-degenerate associative symmetric bilinear forms of some commutative Novikov algebras, such as $(A6)$, $(A7)$, $(A8)$, $(A9)$.
(4) There are non-commutative quadratic Novikov algebras in higher dimensions.

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